Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 4, pp. 527–536, 2008

SOLVABILITITY OF A MODEL PROBLEM OF HEAT AND MASS TRANSFER IN THAWING SNOW

A. A. Papin

UDC 532.546

A model problem of the motion of water and air in thawing snow is examined using the Masket–Leverett equations of two-phase filtration. The theorem of existence of a self-similar solution is proved.

Key words: *two-phase filtration, self-similar solutions, heat and mass transfer.*

Introduction. Various snow cover models are used in the solution of problems of avalanche motion [1], the contribution of snow cover to watershed runoff [2, 3], and contamination propagation in thawing snow [4, 5]. The problem of heat and mass transfer in thawing snow is important for the prediction of watershed runoff and contamination propagation.

Mathematical models of thawing snow cover are constructed using the general principles of dynamics of a multiphase medium [6]. These models necessarily include phase transitions and use the filtrational approximation; therefore, the basic equations of the model are mass and energy conservation laws and the Darcy law for moving phases [3, 4]. This approach has been used in studies of thermal two-phase filtration [7], dissociation of hydrates neighboring ice in natural layers [8], and heat and mass transfer in freezing through and thawing soils [9].

1. Formulation of the Problem. Snow is treated as a porous medium whose solid skeleton consists of motionless ice particles [3]. During thawing, water and air move jointly in the porous medium. Thawing snow is a three-phase medium consisting of water ($i = 1$), air ($i = 2$), and ice ($i = 3$). The process is described using the equation of conservation of mass for each phase [6], the Masket–Leverett equations of two-phase filtration for water and air [7, 10], and the equation of conservation of energy for thawing snow (ignoring sublimation and mass exchange between water and air) [3, p. 144]:

$$\frac{\partial \rho_i}{\partial t} + \operatorname{div}(\rho_i \boldsymbol{u}_i) = \sum_{j=1}^{3} I_{ji}, \quad i = 1, 2, 3, \quad I_{ji} = -I_{ij}, \quad \sum_{i,j=1}^{3} I_{ij} = 0; \tag{1}$$

$$\boldsymbol{v}_i = -K_0 \frac{k_{0i}}{\mu_i} (\nabla p_i + \rho_i^0 \boldsymbol{g}), \quad i = 1, 2, \quad p_2 - p_1 = p_c(s_1, \theta), \quad \sum_{i=1}^{2} s_i = 1; \tag{2}$$

$$\Big(\sum_{i=1}^{3} \rho_i^0 c_i \alpha_i\Big) \frac{\partial \theta}{\partial t} + \Big(\sum_{i=1}^{2} \rho_i^0 c_i \boldsymbol{v}_i\Big) \nabla \theta = \operatorname{div}(\lambda_{\rm c} \nabla \theta) + \nu \frac{\partial \rho_3^0 \alpha_3}{\partial t}. \tag{3}$$

Here t is time, $\boldsymbol{u}_i$ is the velocity of the ith phase, ρ_i is the reduced density related to the true density ρ_i^0 and volumetric concentration α_i by the formula $\rho_i = \alpha_i \rho_i^0$ (the condition $\sum_{i=1}^{3} \alpha_i = 1$ is a consequence of the definition of ρ_i), I_{ji} is the rate of mass transfer from the jth to the ith component per unit volume in unit of time, $\boldsymbol{v}_i = m s_i \boldsymbol{u}_i$ are the filtration velocities of water and air, m is the snow porosity, s_1 and s_2 are the water and air

---

Altai State University, Barnaul 656049; papin@ab.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 4, pp. 13–23, July–August, 2008. Original article submitted March 31, 2006; revision submitted March 5, 2008.

saturations ($\alpha_1 = ms_1$, $\alpha_2 = ms_2$, and $\alpha_3 = 1 - m$), K_0 is the filtration tensor, k_{0i} are the relative phase permeailities [$k_{0i} = k_{0i}(s_i) \geq 0$ and $k_{0i}|_{s_i=0} = 0$], μ_i is the dynamic viscosity, p_i are the phase pressures, p_c is the capillary pressure, $\boldsymbol{g}$ is the acceleration vector due to gravity, θ is the temperature of the medium ($\theta_i = \theta$, where $i = 1, 2, 3$ [3, 7]), $c_i = \text{const} > 0$ is the constant-volume heat capacity of the ith phase, $\nu = \text{const} > 0$ is the specific heat of ice melting, $\lambda_\text{c} = a_\text{c} + b_\text{c}\rho_\text{c}^2$ is the thermal conductivity of snow, $\rho_\text{c} = \sum\limits_{i=1}^{3} \rho_i^0 \alpha_i$, $a_\text{c} = \text{const} > 0$, and $b_\text{c} = \text{const} > 0$ [3, p. 146].

System (1)–(3) is supplemented with the hypotheses $\boldsymbol{u}_3 = 0$ (the ice particles are motionless, and the structure of ice as a continuous medium is not specified [3]), $I_{12} = 0$, $I_{23} = 0$, $I_{31} = I_{31}(\theta)$, and $\rho_i^0 = \rho_i^0(\theta)$, $i = 1, 2, 3$.

The purpose of this work is to construct an exact solution of system (1)–(3) subject to natural boundary conditions.

2. Simple Solution. We introduce finite values of the temperature θ^-, θ_1 and θ^+. Let $0 < \theta^- < \theta_1 < \theta^+$. We assume that, for all $\theta \in (0, \infty)$, the following relations hold: $\alpha_3(\theta) = 0$ for $\theta \geq \theta^+$, $\alpha_3(\theta) = 1 - m^- - m_1(\theta - \theta_1)$ for $\theta_1 \leq \theta \leq \theta^+$, and $\alpha_3(\theta) = 1 - m^-$ for $\theta \leq \theta_1$. Here $m^- = m(\theta^-) \in (0, 1)$ and $m_1 = (1 - m^-)/(\theta^+ - \theta_1)$ are specified parameters. In addition, it is assumed that the porous medium is homogeneous ($K_0 = \text{const} > 0$), $\rho_i^0 = \text{const} > 0$ (these conditions do not influence the generality of the results) and that, in the coordinate system xyz, the vector $\boldsymbol{g} = (0, 0, -g)$ and the functions included in system (1)–(3) depend on z and t. Eliminating I_{31} from (1), we obtain the system of equations

$$\frac{\partial}{\partial t}\,(ms_1\rho_1^0 + \rho_3^0(1 - m)) + \frac{\partial}{\partial z}\,(\rho_1^0 v_1) = 0; \tag{4}$$

$$\frac{\partial}{\partial t}\,(ms_2\rho_2^0) + \frac{\partial}{\partial z}\,(\rho_2^0 v_2) = 0; \tag{5}$$

$$v_i = -K_0\,\frac{k_{0i}}{\mu_i}\Big(\frac{\partial p_i}{\partial z} - \rho_i^0 g\Big), \quad i = 1, 2, \quad p_2 - p_1 = p_c(s_1, \theta), \quad s_1 + s_2 = 1; \tag{6}$$

$$\Big(\sum_{i=1}^{3} \rho_i^0 c_i \alpha_i\Big)\frac{\partial \theta}{\partial t} + \Big(\sum_{i=1}^{2} \rho_i^0 c_i v_i\Big)\frac{\partial \theta}{\partial z} = \frac{\partial}{\partial z}\Big(\lambda_\text{c}\,\frac{\partial \theta}{\partial z}\Big) - \nu\rho_3^0\,\frac{\partial m}{\partial t}. \tag{7}$$

For system (4)–(7), we consider the following problem: snow occupies a region $(-\infty, ct)$, $t > 0$. At $z = -\infty$, water is absent ($s_1 = 0$ and $v_1 = 0$), air is motionless ($v_2 = 0$), and the temperature $\theta = \theta^-$ is specified (lower than the ice melting point); at $z = ct$, the velocities of water ($v_1 = v_1^+$) and air ($v_2 = v_2^+$) and the air pressure ($p_2 = p^+$) are known and the temperature $\theta = \theta^+$ is specified (equal to the ice melting point). Assuming that all unknown functions depend only on the variable $\xi = z - ct$ (c is an unknown constant), from (4)–(7) we obtain

$$-c\,\frac{d}{d\xi}\,(ms_1\rho_1^0 + \rho_3^0(1 - m)) + \frac{d}{d\xi}\,(\rho_1^0 v_1) = 0; \tag{8}$$

$$-c\,\frac{d}{d\xi}\,(ms_2\rho_2^0) + \frac{d}{d\xi}\,(\rho_2^0 v_2) = 0; \tag{9}$$

$$v_i = -K_0\,\frac{k_{0i}}{\mu_i}\Big(\frac{dp_i}{d\xi} - \rho_i^0 g\Big), \qquad p_2 - p_1 = p_c(s_1, \theta), \qquad s_1 + s_2 = 1; \tag{10}$$

$$\Big(\sum_{i=1}^{3} \rho_i^0 c_i(v_i - c\alpha_i)\Big)\frac{d\theta}{d\xi} - c\nu\rho_3^0\,\frac{dm}{d\xi} = \frac{d}{d\xi}\Big(\lambda_\text{c}\,\frac{d\theta}{d\xi}\Big); \tag{11}$$

$$s_1\Big|_{\xi \to -\infty} = 0, \quad \theta\Big|_{\xi \to -\infty} = \theta^-, \quad \frac{\partial \theta}{\partial \xi}\Big|_{\xi \to -\infty} = 0, \quad v_i\Big|_{\xi \to -\infty} = 0; \tag{12}$$

$$p_2(0) = p^+, \quad \theta(0) = \theta^+, \quad v_i(0) = v_i^+, \quad i = 1, 2. \tag{13}$$

The functions $s_1(\xi)$, $v_i(\xi)$, and $p_i(\xi)$ and the constant c are unknown. Problem (8)–(13) is solved as follows. Integrating Eqs. (8)–(10), we find the constant c and representations for the filtration velocities and temperature (see Secs. 2.1 and 2.2). Using these representations and (10), we obtain an equation for the saturation $s_1(\xi)$ (see Sec. 2.3). The investigation of the solvability of problem for $s_1(\xi)$ is completed by the solution of problem (8)–(13).

2.1. *Determination of Filtration Velocities.* From Eqs. (8) and (9) it follows that

$$\rho_1^0 v_1 - c(ms_1\rho_1^0 + (1-m)\rho_3^0) = A_1 = \text{const}; \tag{14}$$

$$\rho_2^0 v_2 - cms_2\rho_2^0 = A_2 = \text{const}. \tag{15}$$

From (14) and (12), we have $A_1 = -c\rho_3^0(1-m^-)$ and $m^- \equiv m(\theta^-)$, and from (15) and (12), we obtain $A_2 = -c\rho_2^0 m^-$. Considering Eqs. (14) and (15) subject to conditions (13), we obtain the following system of equations for the unknown parameters c, s^+ ($s_1 \equiv s$ and $s_2 \equiv 1-s$):

$$v_1^+ = c(s^+ - (\rho_3^0/\rho_1^0)(1-m^-)), \qquad v_2^+ = c(1 - s^+ - m^-),$$

where $m(\theta^+) = 1$ and $\rho_3^0 < \rho_1^0$. The solution of this system has the following form:

1) $s^+ = 1 - m^-$, $c = \dfrac{v_1^+}{(1-m^-)(1-\rho_3^0/\rho_1^0)} < 0$ at $v_2^+ = 0$, $v_1^+ < 0$;

2) $s^+ = (1-m^-)\dfrac{\rho_3^0}{\rho_1^0}$, $c = \dfrac{v_2^+}{(1-m^-)(1-\rho_3^0/\rho_1^0)} < 0$ at $v_1^+ = 0$, $v_2^+ < 0$;

3) $s^+(\lambda) = \dfrac{1-m^-}{1+\lambda}\Big(\lambda + \dfrac{\rho_3^0}{\rho_1^0}\Big)$, $c = \dfrac{(1+\lambda)v_2^+}{(1-m^-)(1-\rho_3^0/\rho_1^0)} < 0$ at $v_1^+ < 0$, $v_2^+ < 0$, where $\lambda \equiv v_1^+/v_2^+ > 0$.

Using Eqs. (14) and (15), the filtration velocities can be represented as

$$v_1 = cms + c(\rho_3^0/\rho_1^0)(m^- - m), \qquad v_2 = cm(1-s) - cm^-. \tag{16}$$

2.2. *Representation for Temperature.* From (16), we have

$$\sum_{i=1}^{3} c_i\rho_i^0(v_i - c\alpha_i) = c\rho_3^0(1-m)(c_1 - c_3) + A_1c_1 + A_2c_2,$$

therefore, relation (11) leads to

$$c\rho_3^0(c_1 - c_3)\int_0^\theta (1 - m(\zeta))d\zeta + (A_1c_1 + A_2c_2)\theta - \nu\rho_3^0 cm - \lambda_\text{c}\frac{d\theta}{d\xi} = \text{const}.$$

Using conditions (12), we obtain

$$\lambda_\text{c}\frac{d\theta}{d\xi} = c\rho_3^0(c_1 - c_3)M(\theta) + (A_1c_1 + A_2c_2)(\theta - \theta^-) - \nu\rho_3^0 c(m - m^-) \equiv f_1(\theta), \tag{17}$$

where

$$M(\theta) \equiv \int_{\theta^-}^{\theta}(1 - m(\zeta))\,d\zeta = \begin{cases} (1-m^-)(\theta^+/2 - \theta^- + \theta_1/2) = M_1, & \theta \ge \theta^+, \\ (1-m^-)(\theta - \theta^-) - m_1(\theta - \theta_1)^2/2, & \theta_1 \le \theta \le \theta^+, \\ (1-m^-)(\theta - \theta^-), & \theta \le \theta_1. \end{cases}$$

For the solution of Eq. (17), the estimate $\theta(\xi) \ge \theta^-$ is valid for all $\xi \in (-\infty, 0)$. Indeed, let $z(\xi) = \theta^- - \theta(\xi)$ and $z^0(\xi) = \max\{z, 0\}$. By virtue of conditions (12) and (13), we have $z^0|_{\xi=0} = 0$ and $z^0|_{\xi=-\infty} = 0$. In addition, on the set $\{\xi\colon \theta^- > \theta(\xi)\}$, we have the function $f_1(\theta) = B(\theta - \theta^-)$, where $B = -c(\rho_3^0(1-m^-)c_3 + \rho_2^0 m^- c_2) > 0$. In view of the aforesaid, Eq. (17) leads to the equality

$$-\frac{1}{2}(z^0)^2\Big|_{-\infty}^{0} = B\int_{-\infty}^{0}\frac{z}{\lambda_\text{c}}z^0\,d\xi = B\int_{\xi|\theta^- > \theta(\xi)}\frac{1}{\lambda_\text{c}}(z^0)^2\,d\xi = 0.$$

Because $\lambda_\text{c} > 0$, it follows that $\theta(\xi) \ge \theta^-$ for almost all ξ. Therefore, for $\theta \in [\theta^-, \infty)$, the functions $m(\theta)$ and $M(\theta)$ can be considered specified, and

$$f_1(\theta) = \begin{cases} c\rho_3^0(c_1-c_3)M_1 + (A_1c_1+A_2c_2)(\theta-\theta^-) - \nu c\rho_3^0(1-m^-), & \theta(\xi) \ge \theta^+, \\ c\rho_3^0(c_1-c_3)[(1-m^-)(\theta-\theta^-) - m_1(\theta-\theta_1)^2/2] & \\ \quad + (A_1c_1+A_2c_2)(\theta-\theta^-) - \nu c\rho_3^0 m_1(\theta-\theta_1), & \theta_1 \le \theta(\xi) \le \theta^+, \\ (c\rho_3^0(c_1-c_3)(1-m^-) + A_1c_1+A_2c_2)(\theta-\theta^-), & \theta^- \le \theta(\xi) \le \theta_1. \end{cases}$$

This function is continuous at the points $\theta = \theta_1$ and $\theta = \theta^+$; $f_1(\theta^-) = 0$. We set

$$\bar{a} = c\rho_3^0(c_1-c_3)M_1 - \nu c\rho_3^0(1-m^-) - \theta^-(A_1c_1+A_2c_2),$$

$$\bar{b} = A_1c_1 + A_2c_2 = -c(\rho_3^0(1-m^-)c_1 + \rho_2^0 m^- c_2), \qquad b_1 = -cm_1\rho_3^0(c_1-c_3)/2,$$

$$d_1 = c\rho_3^0(c_1-c_3)(1-m^-) + A_1c_1 + A_2c_2 - \nu c\rho_3^0 m_1,$$

$$a_1 = (\theta_1-\theta^-)(c\rho_3^0(c_1-c_3)(1-m^-) + A_1c_1 + A_2c_2),$$

$$b_2 = c\rho_3^0(c_1-c_3)(1-m^-) + A_1c_1 + A_2c_2.$$

Then,

$$f_1(\theta) = \begin{cases} \bar{a} + \bar{b}\theta, & \theta \ge \theta^+, \\ b_1(\theta-\theta_1)^2 + d_1(\theta-\theta_1) + a_1, & \theta_1 \le \theta \le \theta^+, \\ b_2(\theta-\theta^-), & \theta^- \le \theta \le \theta_1. \end{cases}$$

Let these problems satisfy the conditions

$$0 < \rho_2^0 < \rho_3^0 < \rho_1^0 < \infty, \qquad 0 < c_3 < c_1 < c_2 < \infty.$$

From the definition of A_1, A_2, c, and m_1, it follows that $b_2 = -c\rho_3^0 c_1((1-m^-)c_3/c_1 + m^- c_3\rho_2^0/(c_1\rho_1^0)) > 0$, $a_1 = b_2(\theta_1-\theta^-) > 0$, $b_1 > 0$, $\bar{b} > 0$, $d_1 = b_2 - \nu c\rho_3^0 m_1 > b_2 > 0$; therefore, for $\theta \in [\theta^-, \infty)$, the function $f_1(\theta)$ is nonnegative and increases monotonically, in particular, $\theta(\xi) \in [\theta^-, \theta^+]$.

Remark 1. According to [11], for real processes, $c_1 = 4.18$ J/(g·K), $c_2 = 29.2$ J/(g·K), $c_3 = 2.04$ J/(g·K), $\nu = 6009/18$ J/g, $\rho_1^0 = 10^3$ kg/m^3, $\rho_2^0 = 1.2928$ kg/m^3, and $\rho_3^0 = 0.925 \cdot 10^3$ kg/m^3.

The solution of problem (12), (13), (17) can be written as

$$I(\theta) \equiv \int_\theta^{\theta^+} \frac{d\zeta}{f_1(\zeta)} = \int_\xi^0 \frac{d\zeta}{\lambda_c(\zeta)} \equiv \psi(\lambda_c(\xi)), \qquad \theta(\xi) = I^{-1}(\psi(\lambda_c(\xi))). \tag{18}$$

For $\theta \in [\theta_1, \theta^+]$, from (18) we obtain

$$I(\theta) = \int_\theta^{\theta^+} \frac{d\zeta}{b_1(\zeta-\theta_1)^2 + d_1(\zeta-\theta_1) + a_1} = \psi(\lambda_c(\xi)).$$

We note that the condition

$$\nu/c_1 \ge (1 - c_3/c_1)(\theta_1 - \theta^-) \tag{19}$$

is sufficient to reduce the function $f_1(\theta)$ in the interval $[\theta_1, \theta^+]$ to the form $b_1[(\theta-\theta_1+\alpha)^2 - \beta^2]$, $\alpha = d_1/(2b_1)$, $\beta^2 = (d_1^2 - 4a_1b_1)/(4b_1^2)$. Remark 1 implies that $\nu/c_1 > 60$ K and $c_3/c_1 < 1/2$ and, for real processes, $\theta_1 - \theta^- < 2\nu/c_1$, i.e., condition (19) is satisfied, which leads to

$$I(\theta) = \frac{1}{b_1}\int_\theta^{\theta^+} \frac{d\zeta}{(\zeta-\theta_1+\alpha)^2 - \beta^2} = \psi(\lambda_c(\xi)), \qquad \theta \in [\theta_1, \theta^+]. \tag{20}$$

Because $\theta(\xi)$ is monotonic, a point ξ_1 exists such that $\theta(\xi_1) = \theta_1$. Relation (20) leads to the following condition for determining ξ_1: $I(\theta_1) = \psi(\lambda_c(\xi_1))$.

For $\theta \in [\theta^-, \theta_1]$, from (18) we obtain

$$\theta(\xi) = \theta^- + (\theta_1 - \theta^-) \exp\Big(- b_2 \int\limits_{\xi}^{\xi_1} \frac{d\zeta}{\lambda_c(\zeta)} \Big). \tag{21}$$

Thus, for the specified function $\lambda_c(s, \theta)$, representation (18) and its particular cases (20) and (21) define the temperature for all $\xi \in (-\infty, 0)$.

2.3. *Determination of Saturations and Pressure.* From (10) and (16), we have

$$v_1 = cms - c \frac{\rho_3^0}{\rho_1^0} (m - m^-) = -K_0 \frac{k_{01}}{\mu_1} \Big(\frac{dp_1}{d\xi} - \rho_1^0 g \Big),$$

$$v_2 = cm(1 - s) - cm^- = -K_0 \frac{k_{02}}{\mu_2} \Big(\frac{dp_2}{d\xi} - \rho_2^0 g \Big).$$

Eliminating p_1 and p_2 from these relations by using the second equation in (10), we obtain

$$-k_{01} k_{02} K_0 \frac{dp_c}{d\xi} = -\mu_1 v_1 k_{02} + \mu_2 v_2 k_{01} + K_0 k_{01} k_{02} g (\rho_1^0 - \rho_2^0).$$

Following [7], we set $p_c(s, \theta) = p_0(\theta) \gamma(s)$ and $d\gamma/ds \equiv \gamma' < 0$. For $s \in (-\infty, +\infty)$, the relative phase permeabilities k_{0i} are determined as follows: $k_{0i} = 0$ for $s_i \le 1$, $k_{0i} = \bar{k}_{0i}(s_i, \theta) s_i^{n_i}$ for $0 \le s_i \le 1$, and $k_{0i} = \bar{k}_{0i}(1, \theta)$ for $s_i \ge 1$. Here $n_i > 1$. It is assumed that $a(s) = -\gamma' k_{01} k_{02} > 0$ for $s \in (0, 1)$ and $a(s) = 0$ for $s \le 0$ and $s \ge 1$.

Using the temperature equation (17), the equation for the saturation can be written as

$$a_0(s) \frac{ds}{d\xi} = \varphi_1 \varphi_2 \gamma \frac{p_0'}{p_0} \frac{f_1}{\lambda_c} + \frac{1}{p_0} \bar{g} \varphi_1 \varphi_2 + \frac{1}{p_0} |c| m A s - \frac{1}{p_0} |c| (m - m^-) B \equiv f_2(s, \theta). \tag{22}$$

Here $\varphi_i = 0$ for $s_i \le 0$, $\varphi_i = s_i^{n_i}$ for $0 \le s_i \le 1$, and $\varphi_i = 1$ for $s_i \ge 1$,

$$a_0 = -\varphi_1 \varphi_2 \gamma', \quad A = \bar{\mu}_1 \varphi_2 + \bar{\mu}_2 \varphi_1, \quad B = \bar{\mu}_1 (\rho_3^0 / \rho_1^0) \varphi_2 + \bar{\mu}_2 \varphi_1, \quad \bar{g} = g(\rho_1^0 - \rho_2^0),$$

$$p_0' = \frac{dp_0}{d\theta}, \qquad \bar{\mu}_i = \frac{\mu_i}{K_0 \bar{k}_{0i}}, \qquad i = 1, 2.$$

Equation (22) is examined for $\xi < 0$ and the condition $s(0) = s^+$ (see conditions 1–3 in Sec. 2.1), i.e., the Cauchy problem is studied for $s(\xi)$, and the condition $s|_{\xi = -\infty} = 0$ should be proved.

To find the pressures p_1 and p_2, we consider the following equality implied by the Darcy law [the first equations in (10)]:

$$\sum_{i=1}^{2} (\bar{\mu}_i v_i - g \rho_i^0 \varphi_i) = -(\varphi_1 + \varphi_2) \Big(\frac{dp_2}{d\xi} - \frac{\varphi_1}{\varphi_1 + \varphi_2} \frac{dp_c}{d\xi} \Big).$$

We set [7]

$$p(\xi) \equiv p_2(\xi) - p_0(\theta) b(s), \qquad b(s) = \int\limits_0^s \frac{\varphi_1(\zeta) \gamma'(\zeta)}{\varphi_1(\zeta) + \varphi_2(\zeta)} \, d\zeta.$$

Then,

$$\frac{dp}{d\xi} = - \frac{1}{\varphi_1(s) + \varphi_2(s)} \Big[\sum_{i=1}^{2} (\bar{\mu}_i v_i(\xi) - g \rho_i^0 \varphi_i)$$

$$- p_0'(\theta) \frac{f_1}{\lambda_c} \Big(\varphi_1(s) \gamma(s) + b(s) (\varphi_1(s) + \varphi_2(s)) \Big) \Big] \equiv f_3(s, \theta),$$

$$p(0) = p^+ - p_0(\theta^+) b(s^+). \tag{23}$$

Relation (23) implies that

$$p(\xi) = p^+ - p_0(\theta^+)b(s^+) - \int\limits_{\xi}^{0} f_3(s(\zeta), \theta(\zeta))\, d\zeta,$$
$$p_2(\xi) = p(\xi) + p_0(\theta)b(s), \qquad p_1(\xi) = p_2(\xi) - p_c(s(\xi), \theta(\xi)). \tag{24}$$

If the functions $s(\xi)$ and $\theta(\xi)$ are found, the filtration velocities $v_i(\xi)$ and the pressure $p_i(\xi)$ are determined from formulas (16) and (24).

Definition 1. The weak solution of problem (8)–(13) are the functions $\theta(\xi)$, $s(\xi)$, $v_i(\xi)$, and $p_i(\xi)$ and the fixed parameter c if:

1) the function $\theta(\xi)$ has a continuous derivative and satisfies Eq. (17) and the conditions $\theta(0) = \theta^+$, $\theta|_{\xi\to-\infty} = \theta^-$ and $\partial\theta/\partial\xi|_{\xi\to-\infty} = 0$;

2) the function $s(\xi)$ has a continuous derivative with weight $a(s)$ and satisfies Eq. (22) and the conditions $s(0) = s^+$ and $s|_{\xi\to-\infty} = 0$;

3) the functions $v_i(\xi)$ satisfy equalities (16) and the conditions $v_i(0) = v_i^+$, $v_i|_{\xi\to-\infty} = 0$;

4) the functions $p_i(\xi)$ satisfy equalities (24) and the condition $p_2(0) = p_2^+$.

Following [7] (using the notation adopted in that paper), we supplement conditions 1–3 (see Sec. 2.1) and the constraints on the constants ρ_i^0, c_i, and ν (see Sec. 2.2) with the assumption that the functions $\bar\mu_i(s,\theta)$, $\bar k_{0i}(s,\theta)$, $p_0(\theta)$, $\gamma(s)$, and $a_0(s)$ satisfy the following conditions ($\bar\Omega^* = [0,1]\times[\theta^-,\theta^+]$):

(a) $0 < \nu_0^{-1} \le \Big(\mu_i(s,\theta), \bar k_{0i}(s,\theta), p_0(\theta), \Big|\dfrac{d\gamma(s)}{ds}\Big|\Big) \le \nu_0$, $\frac{a_0(s)}{s}\Big|_{s=0} = 0$;

(b) $\Big(\Big\|\dfrac{d\gamma}{ds}\Big\|_{C[0,1]}, \Big\|\dfrac{dp_0}{d\theta}\Big\|_{C[\theta^-,\theta^+]}, \|\mu_i(s,\theta), \bar k_{0i}(s,\theta)\|_{C(\bar\Omega^*)}\Big) \le \nu_0$.

Theorem 1. *If the conditions* (*a*) *and* (*b*) *and the condition* $s^+ \in (0,1]$ *are satisfied, at least one weak solution of problem* (8)–(13) *exists. This solution* (*in addition to the definition*) *has the properties*

$$0 \le s(\xi) \le 1, \qquad \theta^- \le \theta(\xi) \le \theta^+, \qquad c = \frac{(1+\lambda)v_2^+}{(1-m^-)(1-\rho_3^0/\rho_1^0)} < 0.$$

A point $\xi_* \in (-\infty, \xi_1]$ *exist such that* $s(\xi) = 0$ *for all* $\xi \le \xi_*$.

To prove the theorem, it is sufficient to establish the solvability [see the definition of the weak solution for $s(\xi)$ and $\theta(\xi)$ of the problem]

$$a_0(s)\,\frac{ds}{d\xi} = f_2(s,\theta), \quad \frac{d\theta}{d\xi} = \frac{f_1(\theta)}{\lambda_c(s,\theta)}, \quad \xi < 0, \quad s(0) = s^+, \quad \theta(0) = \theta^+ \tag{25}$$

and to show that $s(\xi) \equiv 0$ for $\xi \le \xi_*$. Then, all the conditions included in the definition of the weak solution of problem (8)–(13) will be satisfied.

Let $\varepsilon \in (0,1)$ and $a_\varepsilon(s) \equiv a_0(s) + \varepsilon > 0$. For $\xi < 0$, instead of (25) we consider the problem

$$a_\varepsilon(s^\varepsilon)\,\frac{ds^\varepsilon}{d\xi} = f_2(s^\varepsilon,\theta^\varepsilon), \quad \frac{d\theta^\varepsilon}{d\xi} = \frac{f_1(\theta^\varepsilon)}{\lambda_c(s^\varepsilon,\theta^\varepsilon)}, \quad s^\varepsilon(0) = s^+, \quad \theta(0) = \theta^+. \tag{26}$$

The local solvability of problem (26) for each $\varepsilon > 0$ follows from the well-known results [12, p. 21]. We obtain estimates of $s^\varepsilon(\xi)$ and $\theta^\varepsilon(\xi)$ which are uniform in ε. [We note that the results obtained in Sec. 2.2 for $\theta(\xi)$ are valid for $\theta^\varepsilon(\xi)$.]

The following statement specifies the properties of the functions A and B included in the right side of Eq. (22).

Lemma 1. *If* $n_1 > 1$, $n_2 > 1$, $\alpha > 0$, *and* $0 \le x \le 1$, *then*

$$\min(1,\alpha,\pi_\mu) \le \pi(x) \equiv \alpha x^{n_1} + (1-x)^{n_2} \le \max(1,\alpha),$$

where $\pi_\mu = \alpha(1+\beta)^{-n_1/\mu} + \beta^{n_2}(1+\beta)^{-n_2}$ *for* $\mu \le 1$ *and* $\pi_\mu = \alpha(1+\beta)^{-n_1} + \beta^{n_2}(1+\beta)^{-n_2\mu}$ *for* $\mu \ge 1$; $\mu = (n_1-1)/(n_2-1)$; $\beta = (n_1\alpha/n_2)^{1/(n_2-1)}$.

Proof. We have $\pi(1) = \alpha$, $\pi(0) = 1$, and $\pi'' = \alpha n_1(n_1-1)x^{n_1-2} + n_2(n_2-1)(1-x)^{n_2-2} > 0$, i.e., at the local minimum point x_*, we have $x_* + \beta x_*^\mu = 1$. It is clear that $x_* \in (0,1)$. Moreover, if $\mu = 1$, then

$x_* = 1/(1+\beta)$. For $\mu < 1$, we have $x_*(1+\beta) \le 1$ and $1 \le x_*^\mu(1+\beta)$, i.e., $(1+\beta)^{-1} \le x_*^\mu \le (1+\beta)^{-\mu}$. For $\mu > 1$, we have $x_*^\mu(1+\beta) \le 1$ and $1 \le x_*(1+\beta)$, i.e., $(1+\beta)^{-\mu} \le x_*^\mu \le (1+\beta)^{-1}$. Therefore, $\pi(x_*) = \alpha x_*^{n_1} + \beta^{n_2} x_*^{n_2\mu}$, i.e., $\pi(x_*) = \alpha(1+\beta)^{-n_1} + \beta^{n_2}(1+\beta)^{-n_2}$ for $\mu = 1$, $\pi(x_*) \ge \alpha(1+\beta)^{-n_1/\mu} + \beta^{n_2}(1+\beta)^{-n_2}$ for $\mu < 1$, and $\pi(x_*) \ge \alpha(1+\beta)^{-n_1} + \beta^{n_2}(1+\beta)^{-n_2\mu}$ for $\mu > 1$, which completes the proof.

Lemma 2. *If $s^\varepsilon(\xi)$ is a solution of problem* (26) *and $s^+ \in [0,1]$, then $0 \le s^\varepsilon(\xi) \le 1$.*

Proof. From (26) we obtain

$$\frac{ds^\varepsilon}{d\xi} - Rs^\varepsilon = -Q,$$

where

$$R = \frac{1}{p_0 a_\varepsilon}\Big(|c|mA + \bar{g}\,\frac{\varphi_1(s^\varepsilon)}{s^\varepsilon}\varphi_2 + \frac{\varphi_1(s^\varepsilon)}{s^\varepsilon}\varphi_2\gamma\,\frac{f_1}{\lambda_c}(p_0')^+\Big),$$

$$Q = \frac{1}{p_0 a_\varepsilon}\Big(|c|(m-m^-)B + \varphi_1\varphi_2\gamma\,\frac{f_1}{\lambda_c}(p_0')^-\Big),$$

$$(p_0')^+ = \max(0, p_0'), \qquad (p_0')^- = -\min(0, p_0'), \qquad p_0' = (p_0')^+ - (p_0')^-.$$

By virtue of Lemma 1, we have $A > 0$ and $B > 0$ and, hence, $R > 0$ and $Q \geqslant 0$. Therefore,

$$s^\varepsilon(\xi) = \Big(s^+ + \int_\xi^0 Q(x)\exp\Big(-\int_0^x R(\zeta)\,d\zeta\Big)dx\Big)\exp\Big(\int_0^\xi R(\zeta)\,d\zeta\Big) \ge 0.$$

For the function $1 - s^\varepsilon(\xi)$, from (26) we obtain

$$\frac{d(1-s^\varepsilon)}{d\xi} - R_1(1-s^\varepsilon) = -Q_1,$$

where

$$R_1 = \frac{1}{p_0 a_\varepsilon}\Big(|c|mA + \frac{\varphi_2(s^\varepsilon)}{1-s^\varepsilon}\,\varphi_1\gamma\,\frac{f_1}{\lambda_c}\,(p_0')^-\Big) > 0,$$

$$Q_1 = \frac{1}{p_0 a_\varepsilon}\Big[|c|\Big(m\bar{\mu}_1\varphi_2\Big(1 - \frac{\rho_3^0}{\rho_1^0}\Big) + m^-B\Big) + \bar{g}\varphi_1\varphi_2 + \varphi_1\varphi_2\gamma\,\frac{f_1}{\lambda_c}\,(p_0')^+\Big] > 0.$$

Then,

$$1 - s^\varepsilon(\xi) = \Big(1 - s^+ + \int_\xi^0 Q_1(x)\exp\Big(-\int_0^x R_1(\zeta)\,d\zeta\Big)dx\Big)\exp\Big(\int_0^\xi R_1(\zeta)\,d\zeta\Big) \ge 0,$$

which completes the proof.

We set $v^\varepsilon \equiv \int_0^{s^\varepsilon} a_\varepsilon(\zeta)\,d\zeta$. Then, $dv^\varepsilon/ds^\varepsilon = a_0(s^\varepsilon) + \varepsilon > 0$ and $s^\varepsilon = s(v^\varepsilon)$. We consider the problem

$$\frac{dv^\varepsilon}{d\xi} = f_2(s(v^\varepsilon), \theta^\varepsilon), \qquad \frac{d\theta^\varepsilon}{d\xi} = \frac{f_1(\theta^\varepsilon)}{\lambda_c(s(v^\varepsilon), \theta^\varepsilon)}, \qquad \xi < 0,$$

$$v^\varepsilon(0) = \int_0^{s^+} a_\varepsilon(\zeta)\,d\zeta \equiv v_\varepsilon^+, \qquad \theta^\varepsilon(0) = \theta^+. \tag{27}$$

By virtue of the properties of $\bar{\mu}_1$, $\bar{\mu}_2$, and p_0 and Lemma 2, the functions f_2 and f_1/λ_c are continuous in the variables ξ, s^ε, and θ^ε and are uniformly bounded in ε. Therefore, the solution of problem (27) exists and can be extended to any finite interval [12, p. 24]. In particular, for any point $\xi \in [\xi_1, 0]$, we have $|dv^\varepsilon/d\xi| \le C(\nu_0, n_1, n_2, g_\mu)$. Thus, the family of functions $\{v^\varepsilon(\xi)\}$ is equivalently continuous and is uniformly bounded for all $\xi \in [\xi_1, 0]$. The same property for $\{\theta^\varepsilon(\xi)\}$ follows from the second equation in (27), and for $\{s^\varepsilon(\xi)\}$, it follows from the inequality

$|s^\varepsilon(\xi_1) - s^\varepsilon(\xi_2)| \le 4^{-\alpha}\alpha K|v^\varepsilon(\xi_1) - v^\varepsilon(\xi_2)|^\alpha$ obtained in [7, p. 95] for $a_0(s) \ge K(s(1-s))^\varkappa$, $\varkappa = \text{const} > 0$, and $K = \text{const} > 0$.

By virtue of Arzela's theorem, from the sequences $\{v^\varepsilon(\xi)\}$, $\{s^\varepsilon(\xi)\}$, and $\{\theta^\varepsilon(\xi)\}$, it is possible to choose subsequences that converge uniformly to $v(\xi)$, $s(\xi)$, and $\theta(\xi)$. Because of the continuity of $f_2(s,\theta)$ and $f_1(\theta)/\lambda_c(s,\theta)$, in the equalities

$$v_\varepsilon^+ - v^\varepsilon(\xi) = \int_\xi^0 f_2(s^\varepsilon(\zeta), \theta^\varepsilon(\zeta))\, d\zeta, \qquad \theta^+ - \theta^\varepsilon(\xi) = \int_\xi^0 \frac{f_1(\theta^\varepsilon(\zeta))}{\lambda_c(s^\varepsilon(\zeta), \theta^\varepsilon(\zeta))}\, d\zeta$$

it is possible to pass to the limit as $\varepsilon \to +0$. Thus, the limiting functions satisfy the corresponding integral equations, i.e., they are a solution of problem (25).

The function $s(\xi)$ is continuous in the interval $[\xi_1, 0]$, and, hence, a value $s^1 \equiv s(\xi_1) \in [0,1]$ exists. Therefore, one can consider the problem

$$a_0(s)\,\frac{ds}{d\xi} = f_2(s,\theta), \quad \frac{d\theta}{d\xi} = \frac{f_1(\theta)}{\lambda_c(s,\theta)}, \quad \xi < \xi_1, \quad s(\xi_1) = s^1, \quad \theta(\xi_1) = \theta_1, \tag{28}$$

where

$$f_2 = \varphi_1\varphi_2\gamma\,\frac{p_0'}{p_0}\,\frac{f_1}{\lambda_c} + \frac{1}{p_0}\,\bar g\varphi_1\varphi_2 + \frac{1}{p_0}\,|c|m^- As.$$

Remark 2. The following statement is valid: if $s^+ > 0$, then $s(\xi) > 0$ at any internal point of the interval $[\xi_1, 0]$. To prove this, we consider system (26) in the interval $I_\tau = [\xi_\tau, 0]$, where $\xi_\tau = \xi_1 + \tau$ ($\tau > 0$ is a small number). In this interval,

$$\min_{\xi\in I_\tau}\Big(\frac{1}{p_0}\,|c|(m - m^-)B\Big) \equiv B_\tau > 0$$

by virtue of the definition of $m(\theta)$ and the monotonicity of $\theta(\xi)$. Then, on the set $A_\delta = \{\xi \in I_\tau, s^\varepsilon < \delta\}$, the function

$$-f_2(s^\varepsilon, \theta^\varepsilon) \ge B_\tau - \delta\max_{\xi\in I_\tau}\Big(\frac{1}{p_0}\,\Big(\varphi_1\varphi_2\gamma|p_0'|\,\frac{f_1}{s\lambda_c} + \bar g\varphi_1\varphi_2\,\frac{1}{s} + |c|mA\Big)\Big) \ge \frac{1}{2}\,B_\tau$$

for $0 < \delta \le \delta_0$, where

$$\delta_0 = \frac{1}{2}\,B_\tau\Big[\max_{\xi\in I_\tau}\Big(\frac{1}{p_0}\Big(\varphi_1\varphi_2\gamma|p_0'|\,\frac{f_1}{s\lambda_c} + \bar g\varphi_1\varphi_2\,\frac{1}{s} + |c|mA\Big)\Big)^{-1}\Big].$$

We consider the function $z^\delta(\xi) = \max\{\delta - s^\varepsilon, 0\}$, where $\delta < \min(s^+, \delta_0)$. We have $z^\delta(0) = 0$. Multiplying the first equation in (26) by $z^\delta(\xi)/a_\varepsilon$ and integrating the resulting equality over ξ from $\xi_0 \in I_\tau$ to zero, we obtain

$$(z^\delta(\xi_0))^2 - 2\int_{\xi_0}^0 \frac{z^\delta f_2}{a_\varepsilon}\, d\zeta = 0.$$

From this it follows that $z^\delta(\xi) = 0$; therefore, $s^\varepsilon(\xi) \ge \delta$ for all $\xi \in I_\tau$ and for any $\varepsilon > 0$. For $\varepsilon \to 0$, we obtain a similar inequality for $s(\xi)$.

Lemma 3. *Let $s(\xi)$ be a solution of problem* (28) *and let the condition $\varphi_1(s)\gamma(s)/s \le \nu_0$ ($s \in [0,1]$) be satisfied. Then, a point $\xi_* \le \xi_1$ exists such that $s(\xi) \equiv 0$ for all $\xi \le \xi_*$. If $s^1 = 0$ and $\varphi_1(s)\gamma(s)/s\Big|_{s=0} = 0$, we have $\xi_* = \xi_1$.*

Proof. Relation (28) implies that

$$\frac{du}{d\xi} + D_1(s,\theta) = D_2(s,\theta), \qquad u = \int_0^s \frac{a_0(\zeta)}{\zeta}\, d\zeta. \tag{29}$$

Here

$$D_1(s,\theta) = \frac{\varphi_1(s)}{s}\,\varphi_2\gamma(p_0')^-\,\frac{f_1}{p_0\lambda_c}, \qquad f_1 = b_2(\theta - \theta^-) \ge 0,$$

$$D_2(s,\theta) = \frac{1}{p_0}\Big(|c|m^- A + \bar{g}\,\frac{\varphi_1}{s}\,\varphi_2 + \frac{\varphi_1(s)}{s}\,\varphi_2\gamma(p_0')^+\,\frac{f_1}{\lambda_c}\Big).$$

According to Lemma 1, we have

$$D_2(s,\theta) \ge |c|m^- A \min_{s,\theta}(\bar{\mu}_1/p_0)\min(1,\alpha,\pi_\mu) \equiv D_2^0 > 0.$$

Here $\alpha = \min\limits_{s,\theta} \bar{\mu}_1/\bar{\mu}_2 > 0$, where $\theta \in [\theta^-, \theta^+]$ and $s \in [0,1]$. Using representation (21) for the function $\theta(\xi) \in [\theta^-, \theta_1]$, we obtain

$$|D_1| \le D_1^0 \exp\Big(-\frac{1}{\lambda_c^+}(\xi_1 - \xi)\Big), \qquad \int\limits_{\xi}^{\xi_1} |D_1(\zeta)|\,d\zeta \le \frac{1}{\lambda_c^+}\,D_1^0,$$

where

$$D_1^0 = \frac{1}{\lambda_c^-}\,b_2(\theta_1 - \theta^-)\max_{s,\theta}\Big(\frac{1}{p_0}\,\frac{\varphi_1(s)}{s}\,\varphi_2\gamma(p_0')^-\Big) \le \frac{1}{\lambda_c^-}\,b_2(\theta_1-\theta^-)\nu_0^3,$$

$$\lambda_c^- = a_c + b_c(\rho_2^0 m^-)^2, \qquad \lambda_c^+ = a_c + b_c(\rho_1^0 m^- + \rho_3^0(1-m^-))^2.$$

Integration of Eq. (29) over ξ from any value ξ to ξ_1 yields

$$u(s(\xi_1)) + D_1^0/\lambda_c^+ \ge D_2^0(\xi_1 - \xi) + u(s(\xi)). \tag{30}$$

Here $u(s(\xi_1)) = 0$ for $s(\xi_1) = 0$ and $u(s(\xi_1)) \le \int\limits_0^1 \frac{a_0(\zeta)}{\zeta}\,d\zeta$ for $s(\xi_1) > 0$. The last integral converges by virtue of the assumptions of Lemma 3; therefore, $u(s(\xi)) \le 0$ for all $\xi \le \xi_*$, where ξ_* satisfies the condition

$$D_2^0\xi_* = D_2^0\xi_1 - \frac{1}{\lambda_c^+}\,D_1^0 - \int\limits_0^1 \frac{a_0(\zeta)}{\zeta}\,d\zeta.$$

Then, the definition of $u(s)$ implies that $u(s(\xi)) \equiv 0$ for $\xi \le \xi_*$.

Let $s(\xi_1) = 0$ and $\varphi_1(s)\gamma(s)/s|_{s=0} = 0$ [in this case, $s(\xi) = 0$ satisfies the first equation in (28)]. If $s(\xi)$ is a solution of (28), then, by virtue of Lemma 2 the functions $u(\xi)$ and $s(\xi)$ are continuous in ξ. We consider a small vicinity of the point ξ_1, assuming that, at the point $\xi = \xi_1 - \delta$, $\delta > 0$, the inequality $s(\xi) > 0$ holds. For $\xi \in [\xi_1 - \delta, \xi_1]$, from Eq. (29) we obtain

$$\frac{du}{d\xi} \ge D_2^0 - \min_{s,\theta}\Big((p_0')^-\,\frac{f_1}{p_0\lambda_c}\Big)\frac{\varphi_1(s)}{s}\,\varphi_2\gamma \ge \frac{1}{2}\,D_2^0$$

due to the appropriate choice of δ. Then, $0 = u(s(\xi_1)) \ge D_2^0\delta/2 + u(s(\xi))$, i.e., $u(s(\xi)) < 0$ and, hence, $s(\xi) = 0$. Repeating this process, in the kth step, we obtain $s(\xi_k) = 0$, $\xi_k = \xi_1 - k\delta$, where $k > 1$. Reaching the value k for which the inequality $D_1^0 \le \lambda_c D_2^0 k\delta$ is valid we use (30) and obtain $s(\xi_k) = 0$ and $\xi \in (-\infty, \xi_1]$. Lemma 3 is proved.

In view of Lemma 3, problem (28) is considered similarly to problem (25).

Thus, functions $s(\xi)$ and $\theta(\xi)$ exists that satisfy the definition of the weak solution of problem (8)–(13). Theorem 1 is proved.

REFERENCES

1. V. Blagovechshenskiy, M. Eglit, and M. Naaim, "The calibration of avalanche mathematical model using field data," *Natur. Hazards*, No. 2, 203–209 (2002).
2. E. A. Anderson, "Hydro 17 — snow model. NWSRES users manual," Part 11.2, NOAA Nat. Weather Service, Silver Spring (1996).
3. L. S. Kuchment, V. N. Demidov, and Yu. G. Motovilov, *River Runoff Formation. Physical and Mathematical Models* [in Russian], Nauka, Moscow (1983).
4. A. C. Fowler, *An Introduction to Mathematical Modeling*, Oxford Univ., Oxford (2002).

5. K. Rankinen, T. Karvonen, and D. Butterfield, "A simple model for predicting soil temperature in snow-cover and seasonally frozen soil: model description and testing," *Hydrology Earth System Sci.*, No. 8, 706–716 (2004).
6. R. I. Nigmatulin, *Dynamics of Multiphase Media*, Part 1, Hemisphere, New York (1991).
7. B. T. Zhumagulov and V. N. Monakhov, *Hydrodynamics of Oil Production* [in Russian], Kazakh State Institute of Scientific-Engineering Information, Almaty (2001).
8. G. G. Tsypkin, "Mathematical model for the dissociation of gas hydrates coexisting with gas in fields," *Dokl. Ross. Akad. Nauk*, **381**, No. 1, 56–59 (2001).
9. V. I. Vasil'ev, A. M. Maksimov, E. E. Petrov, and G. G. Tsypkin, *Heat and Mass Transfer in Freezing and Thawing Soils* [in Russian], Nauka, Moscow (1997).
10. J. Bear, D. Zaslavsky, and S. Irmay, *Physical Principles of Water Percolation and Seepage,* UNESCO Arid Zone Research, No. 29, Paris (1968).
11. *V. A. Rabinovich and Z. Ya. Khavin*, *Brief Chemical Dictionary* [in Russian], Khimiya, Moscow (1978).
12. P. Hartman, *Ordinary Differential Equations*, Wiley, New York (1964).